Dense flows of bidisperse assemblies of disks down an inclined plane

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Using discrete numerical simulations, we have studied the flow down a rough inclined plane of a bidisperse assembly of frictional cohesionless disks. Our study focuses on steady uniform flows, once a stable segregation has developed inside the flowing layer. The material is segregated in three layers: a basal layer (small grains), a superficial layer (large grains), and a mixed layer in the center, so that the average diameter of the grains increases from the bottom to the top. From the measurement of the profiles of velocity, solid fraction, and stress components, we show that the rheological law of such a polydisperse material may be described by a local friction law, which extends the result obtained for quasimonodisperse granular flows. This law states that the effective friction coefficient depends approximately linearly on a generalized inertial number, taking into account the average diameter of the grains. © *2007 American Institute of Physics.*
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Dense granular flows, involving multiple contacts between grains, are frequent in many natural events and industrial processes. Recent experimental and numerical studies have investigated the rheological behavior in this dense flow regime.[1–4] However, they are restricted to the simple case of spherical, quasimonodisperse, and cohesionless grains without the effect of interstitial fluid. This corresponds to dense and large enough grains (diameter $d \gtrsim 0.1$ mm and mass density $\rho \gtrsim 2000$ kg m^{-3}) immersed in a fluid of low-viscosity-like air. For rigid and quasimonodisperse grains, dimensional analysis shows that the shear rate $\dot{\gamma}$ and the pressure P enter into the problem through the dimensionless *inertial number*,

$$I = \dot{\gamma} d \sqrt{\frac{\rho}{P}}. \qquad (1)$$

Then, the *friction law* in the dense flow regime ($10^{-3} \lesssim I \lesssim 0.3$) may be simply expressed by the approximately linear dependency of the effective friction coefficient $\mu^* = \tau/P$ (with the shear stress τ) as a function of I,

$$\mu^* \approx \phi + bI, \qquad (2)$$

where the parameters ϕ and b are peculiar to the properties of the flowing grains.

If dense flows of quasimonodisperse grains are now well understood, lots of real granular flows such as snow avalanches,[5] pyroclastic flows,[6] or fresh concrete flows[7] are constituted of grains of various sizes. Several studies have characterized the segregation of polydisperse mixture during the flows,[6,8–11] but its effect on rheological behavior is still largely ignored, except for more dilute flows dominated by binary collisions.[12–14] How are flows of quasimonodisperse grains affected when introducing larger grains? What kind of constitutive law can describe dense flows of polydisperse mixtures?

Here we shall provide a first step in the understanding of the rheological behavior of polydisperse granular flows thanks to the study of a system that is as simple as possible:
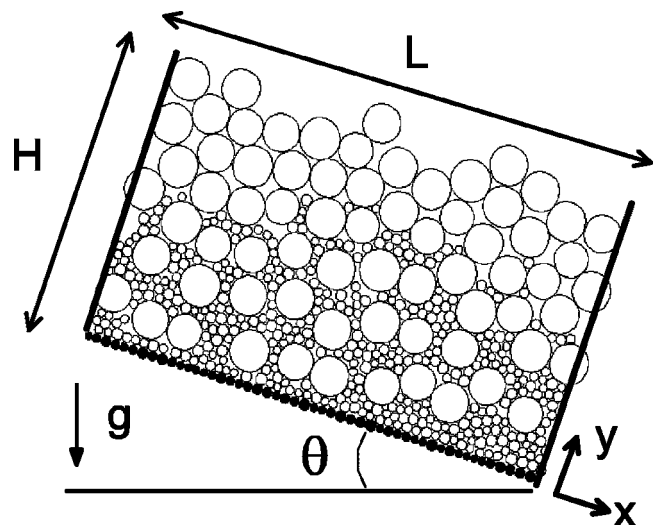

FIG. 1. Bidisperse granular flow down a rough inclined plane ($D_r = 4$, $S_r = 3/4$, $\theta = 17°$, $H \approx 30 d_s$).
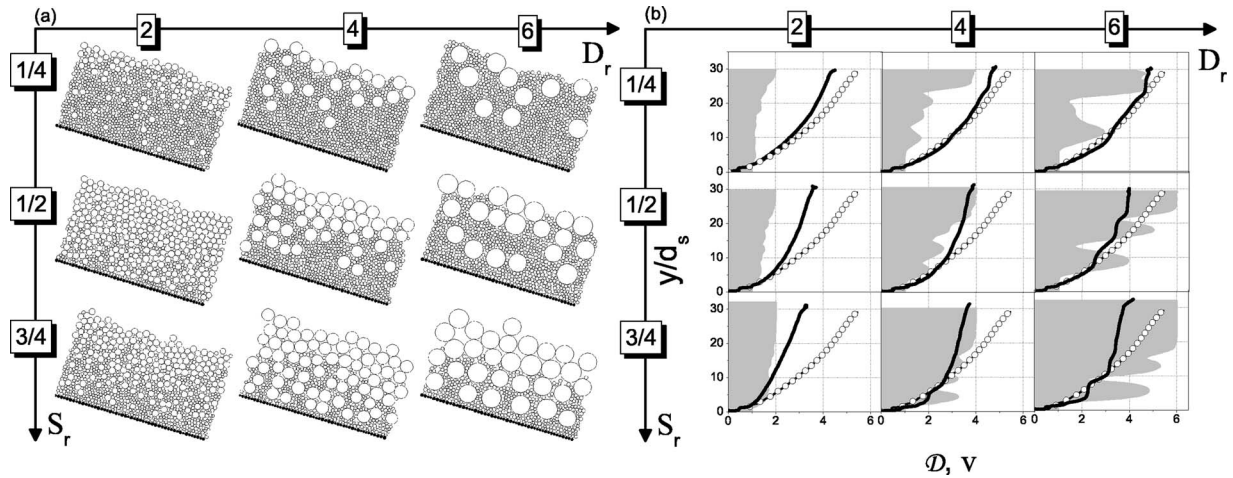
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FIG. 2. Steady and uniform bidisperse flows ($\theta = 17°$, $H \approx 30d_s$): (a) picture of simulations, (b) comparison of the velocity profiles $V(y)$ between bidisperse mixtures (—) and quasimonodisperse small grains ($\bigcirc$); mean diameter profiles $\mathcal{D}(y)$ (gray).

a bidisperse assembly of grains flowing down a rough inclined plane (Fig. 1).

The flows are performed using a standard molecular-dynamics method fully described in Refs. 3, 15, and 16, which gives direct access to solid fraction, velocity, and stresses within the flow. We have chosen a two-dimensional system in order to keep a low number of grains (i.e., a low computational time), which allows us to explore a large range of various parameters.

The granular material is a bidisperse assembly of 500 to 2000 disks with the same mass density ρ: n_s small disks of average diameter d_s and n_l large disks of average diameter d_l. We introduce a small polydispersity ($\pm 20\%$) around each average diameter to prevent crystallization. Two dimensionless parameters characterize the mixture: the size ratio $D_r = d_l / d_s$ and the areal proportion of large grains $S_r = (n_l d_l^2)/(n_l d_l^2 + n_s d_s^2)$. We have studied the following mixtures: $D_r = \{2; 3; 4; 6; 8\}$ and $S_r = \{0; 1/4; 1/2; 3/4\}$.

The grains interact through direct viscoelastic and frictional contacts. The normal contact force, a function of the normal deflection (or apparent interpenetration) h, is the sum of a linear (unilateral) elastic repulsion $N^e = k_n h$ with a normal elastic stiffness coefficient k_n and of a normal viscous force N^v that dissipates energy during collisions: $N^v = \zeta \dot{h}$. ζ is related to the restitution coefficient e in a binary collision between small grains of mass m, $\zeta = \sqrt{2 m k_n}$ $(-\ln e)/\sqrt{\pi^2 + \ln^2 e}$. Friction between grains is described by a Coulomb condition $|T| \leq \mu N^e$, where μ is the coefficient of friction between grains, and the tangential contact force T is related to the *elastic part* δ^e of the relative tangential displacement, $T = k_t \delta^e$, with a tangential elastic stiffness coefficient k_t. δ^e is defined by $\dot{\delta^e} = 0$ if $|T| = \mu N^e$ and $T V^T \geq 0$ (*sliding contact*), $\dot{\delta^e} = V^T$ otherwise (*sticking contact*), where V^T is the tangential relative velocity at the contact point. The stiffness k_n is set large enough to keep the deformation h of the grains smaller than 10^{-3}. In this rigid grain limit, it has been shown that the rheology does not depend on k_n, μ (except for frictionless grains, $\mu = 0$) and e (except for the extreme values $e = 0$ and 1).[3] It was also shown that k_t / k_n has a very

small influence.[15,17] Consequently, the simulations were done with $\mu = 0.4$, $e = 0.1$, and $k_t / k_n = 0.5$. In the following, we use the length d_s, time $\sqrt{d_s / g}$, and velocity $\sqrt{g d_s}$ scales.

We consider a dense layer of height H flowing down a rough inclined plane (inclination θ) under the influence of gravity g (Fig. 1). Along the flow direction x, periodic boundary conditions are applied and the length of the cells is $50 d_s$. The roughness is made of contiguous grains sharing the characteristics of the small flowing grains. We have studied the following situations: $12° \leq \theta \leq 30°$ and $10 \leq H/d_s \leq 50$.

Grains are usually randomly deposited without contact and without velocity with a low solid fraction around 0.5. Then the prescribed slope is applied. After a sufficient amount of time, the flowing layer may reach a steady state characterized by a constant time-averaged velocity profile, solid fraction, and stress tensor. In a large range and slope $12° \leq \theta \leq 25°$ (that depends slightly on the mixture), previous quantities are constant along the flow direction. In the following, we focus on such steady and uniform flows. We measure their profiles of solid fraction, velocity, and stress tensor along the transverse direction y, averaging over time and flow direction.

Figure 2(a) shows pictures of such uniform and steady flows for different mixtures. For each mixture, the total solid fraction profile $\nu(y)$ is approximately constant, with an average value $\nu \simeq 0.8$ that depends very slightly on the composition of the mixture. Accordingly, we check that the pressure and shear stress verify

$$\begin{pmatrix} P(y) \\ \tau(y) \end{pmatrix} = \rho g \nu (H - y) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \tag{3}$$

so that $\mu^* = \tan \theta \simeq \theta$.

Although the total solid fraction is constant, the grains are organized in three layers in the flows [Fig. 2(a)]: a layer of small grains near the rough bed, a layer of large grains near the free surface, and a mixed layer at the center. Those segregated states are stable in time and we checked that they are also independent of both initial configuration and velocity of the grains. As a way to characterize those segregated
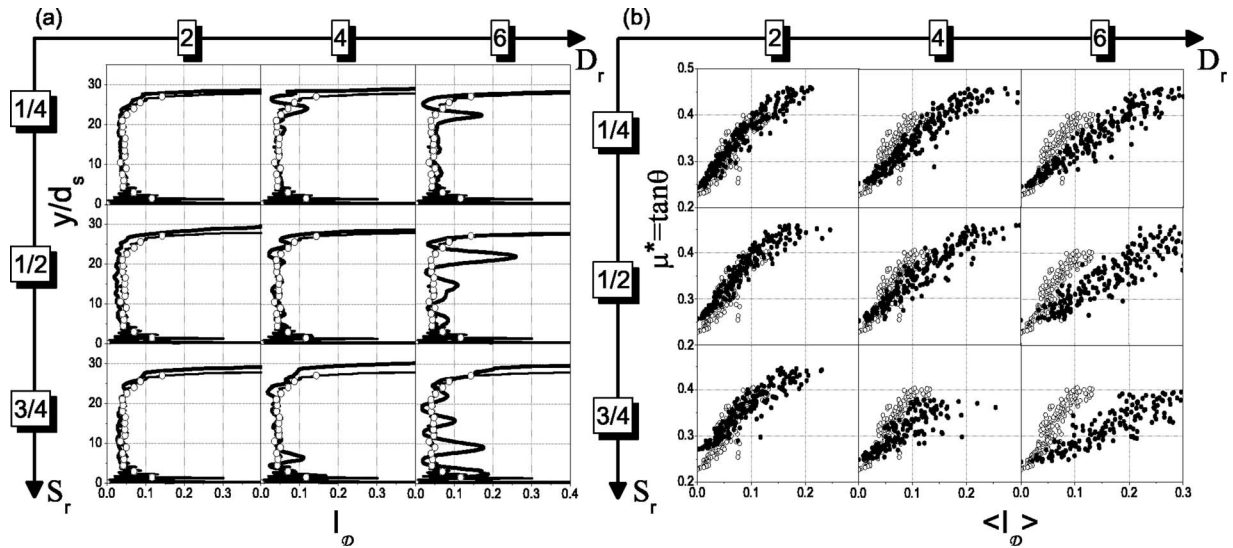
FIG. 3. (a) Profiles of the generalized inertial number $I_\mathcal{D}(y)$ ($\theta=17°$, $H\approx30d_s$) for various bidisperse mixture (—), as compared with quasimonodisperse systems ($\bigcirc$). (b) Local friction law $\mu^*(I_\mathcal{D})$ for various bidisperse mixtures ($\bullet$), as compared with quasimonodisperse systems ($\bigcirc$).

states, we introduce the dimensionless number $\mathcal{D}$, which is the ratio between the mass average diameter of the mixture and the diameter of small grains,

$$\mathcal{D}(y) = \frac{\nu_s(y) + D_r\nu_l(y)}{\nu(y)}. \quad (4)$$

$\nu_s(y)$ and $\nu_l(y)$ are, respectively, the partial solid fraction of the small and large grains $[\nu(y)=\nu_s(y)+\nu_l(y)]$. For each of the mixtures, $\mathcal{D}$ increases from 1 in the small grain layer near the bottom up to D_r in the large grain layer near the top [Fig. 2(b)].

Figure 2(b) also compares the dimensionless velocity profiles $V(y)=v_x(y)/\sqrt{gd_s}$ of bidisperse flows with the quasimonodisperse equivalent flow one (same inclination $\theta=17°$ and thickness $H\approx30d_s$, but with $S_r=0$).

When applied to a steady and uniform dense flow of quasimonodisperse grains down a rough inclined plane, the rheological law (2) predicts that the profile of inertial number $I(y)$ is constant along the depth of the flowing layer, with a value $I\approx(\theta-\phi)/b$. This provides the following dependencies of the shear rate ($\cos\theta\simeq1$):

$$\dot{\gamma}(y) \simeq \frac{\sqrt{\nu g}(\theta-\phi)}{bd}\sqrt{H-y}, \quad (5)$$

that is, a Bagnold-type velocity profile. This prediction has been checked for thick enough layers,[2] but deviations have been observed for thin layers where the velocity profile appears more linear.[15,18]

In the presence of large grains, the shear rate $\dot{\gamma}=dv_x/dy$ decreases systematically in the upper layer of the flow. The thickness of this layer increases with S_r. Moreover, for a large enough proportion of large grains ($S_r\geq3/4$) and for a large enough size ratio ($D_r\geq4$), we also observe a strong increase of the shear rate near the bed. For such a mixture, the profiles of $\mathcal{D}$ reveal the presence of large grains

in the first bottom layers. We think that this increase of shear rate is the consequence of a less efficient trapping of the large grains by the roughness.[19]

The Bagnold scaling (5) highlights the dependency of the shear rate with the size of the grains: $\dot{\gamma}\propto1/d$. Qualitatively, this predicts a decrease of the shear rate in the top layer made of large grains, with a ratio equal to the size ratio D_r. This simple result was quantitatively shown in Ref. 20, on the basis of a simple two-layer model (with complete segregation).

However, this analysis is not sufficient to describe the rheology of the mixed layer, where the inertial number I is no longer constant. But if we consider a new definition of the inertial number, introducing the mean diameter $\mathcal{D}d_s$,

$$I_\mathcal{D} = \dot{\gamma}\mathcal{D}d_s\sqrt{\frac{\rho_p}{P}}, \quad (6)$$

then the profile of this generalized inertial number $I_\mathcal{D}$ becomes constant, except in the bottom and top layers, and except for oscillations in the case of a small number of layers of large grains [see Fig. 3(a)]. This makes $\mathcal{D}d_s$ a good candidate to describe the shear state of the mixed layer. $I_\mathcal{D}$ appears as an extension of I for a bidisperse mixture (it is equal to I for quasimonodisperse grains). Figure 3(a) compares the profiles of $I_\mathcal{D}$ between bidisperse and quasimonodisperse granular flows, and reveals good agreement. I diverges in the top layer because the pressure tends to 0 when approaching the free surface, and increases near the bed possibly due to the structure of the grains in layers. Consequently, we keep the average value of $\langle I_\mathcal{D}\rangle$ in the central part of the flowing layer ($5\lesssim y/d_s\lesssim H-5$).

As a way to measure the friction law of bidisperse mixtures, we start from a steady and uniform flow, and change the inclination θ slowly enough that the flow can be considered steady at each time step. Even with the large dispersion of the data due to the lack of time average (each point cor-

responds to one configuration), Fig. 3(b) reveals that the friction law (2) may be generalized to bidisperse mixtures,

$$\mu^* = \phi(D_r, S_r) + b(D_r, S_r)I_{\mathcal{D}}, \tag{7}$$

that is, an approximately linear increase of the effective friction coefficient as a function of the generalized inertial number $I_{\mathcal{D}}$. However, the two parameters ϕ and b depend on the composition of the mixture $\{D_r; S_r\}$. The decrease of b is especially noticeable for large D_r and S_r, where the granular flow is made of a small number of layers of large grains sliding on a rough bed made of small grains, so that this might be mostly an effect of the interaction with the wall. This might also be an effect of the small size of the system in this limit.

Using discrete numerical simulations, we have studied the influence of bidispersity on granular flows down a rough inclined plane. Steady uniform flows reach a stable segregated state made of three layers: a basal layer made of small grains, a superficial layer made of large grains, and a mixed layer in the center. Large grains decrease the shear rate in the upper layer but favor sliding at the bed. From the measurement of the profiles of velocity, solid fraction, and stress components, we show that the rheological law of such a polydisperse material may be described by a local friction law, which extends the result obtained for quasimonodisperse granular flows. This law states that the effective friction coefficient depends approximately linearly on a generalized inertial number, taking into account the average diameter of the grains.

This work provides a first step in the understanding of the rheological behavior of dense polydisperse granular flows, which suggests several complementary studies. The interaction of the mixture with the roughness of the bed requires a specific study, by varying this roughness,[19] and considering thicker layers. The study of homogeneous plane shear flows without walls[21] would avoid the heterogeneity of the flow down an inclined plane. The extension to three-dimensional polydisperse granular materials and/or to a larger distribution of grain size is clearly needed for comparison with real granular flows. However, the strong increase of the number of grains asks for larger computational power. Then, a coupled model for the segregation itself,[6,8–11] able to predict the profile $\mathcal{D}(y)$, would provide a complete description of the polydisperse flow down a rough plane.

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